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Quotients on the Sato Grassmannian and the moduli of vector bundles*

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Abstract

It is shown that there exists a geometric quotient of the subscheme of stable points of $\mathrm{Gr}(\mathbb{C}((z))^{\oplus r})$ under the action of $\mathrm{Sl}(r, \mathbb{C})$. The consequences in terms of vector bundles on an algebraic curve are studied.

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1. Introduction

In this paper, it is shown that there exists a geometric quotient of the subscheme of stable points of $\mathrm{Gr}(\mathbb{C}((z))^{\oplus r})$ under the action of $\mathrm{Sl}(r, \mathbb{C})$ following GIT techniques.

It is worth recalling that Sato Grassmannians have shown up as a fruitful tool for a variety of problems, e.g. integrable systems [AHP, DJKM, SS], moduli spaces [AMP, M, SW] and field theories [KNTY, MP, Wi]. Because of the existence of *symmetries* one is led to wonder about the existence of quotients. A standard (and powerful) procedure to carry out such a study is the *geometric invariant theory* [MFK]. We hope to apply our results to some issues arising in CFT on Riemann surfaces. These results are also closely related to the study of algebraic solutions of the multicomponent KP hierarchy as well as to the connection between vector bundles and Yang–Mills connections on Riemann surfaces. Due to the length restriction we cannot be more explicit on these relationships; however, the *physical reader* is directed to chapter 8 of [MFK], and [KNTY, Wi] (and references therein).

However, the main obstacle when applying GIT to our situation comes from the fact that Sato Grassmannians are not schemes of finite type. The second section is devoted to providing a way to overcome this problem and shows how Sato Grassmannians can be constructed from the schemes of finite type (theorem 2.1). Section 3, which recalls the notion of stability from

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[CMP], shows a similar result for the subscheme of the Grassmannian consisting of stable points and finishes with the existence of the geometric quotient (theorem 3.4). Here, due to the restrictions on length and for the sake of clarity, we have only dealt with stability but a similar study can be carried out for semistability. Finally, as an application of our results, we use the Krichever map to study the relation between our results and the well-known results for the case of vector bundles on algebraic curves.

Let us finish this introduction by pointing out a future line of research. Once the quotient by $\mathrm{Sl}(r, \mathbb{C})$ has been constructed, one should develop a theory of stability under $\mathrm{Sl}(r, \mathbb{C}[[z]])$ and discuss the possible quotients. As an application, one should study the space of invariants of $H^0(\mathrm{Gr}(\mathbb{C}((z))), \mathrm{Det}^*)$ (Det being the determinant line bundle) since these spaces are closely related to the spaces of conformal blocks in conformal field theory.

2. Preliminaries on infinite Grassmannians

In this section, we recall some definitions and results about infinite Grassmannians. For more details on this subject we point the reader to [AMP] and [CMP].

Let us begin with the definition of infinite Grassmannians. Let V be a $\mathbb{C}$-vector space and V_+ a subspace of V. We say that a subspace $A \subset V$ is commensurable with V_+ when $\dim_{\mathbb{C}}(A + V_+)/(A \cap V_+) < \infty$ and we denote this by $A \sim V_+$. The pair (V, V_+) is assumed to satisfy

- $\bigcap\limits_{A \sim V_+} A = (0)$,
- $V = \varprojlim\limits_{A \sim V_+} V/A$.

The infinite Grassmannian $\mathrm{Gr}(V, V_+)$ (briefly $\mathrm{Gr}(V)$ if we fix V_+) is the $\mathbb{C}$-scheme whose rational points are

$$\mathrm{Gr}(V) = \left\{ \begin{matrix} \mathbb{C}\text{-subspaces } F \subset V \text{ such that} \\ \dim_{\mathbb{C}} V/(V_+ + F) < \infty, \qquad \dim_{\mathbb{C}} F \cap V_+ < \infty \end{matrix} \right\}.$$

The index or characteristic of $F \in \mathrm{Gr}(V)$

$$\chi(F) = \dim_{\mathbb{C}}(F \cap V_+) - \dim_{\mathbb{C}} \left(\frac{V}{F + V_+} \right)$$

is locally constant as a function of F. If $\mathrm{Gr}^{\chi}(V)$ denotes the set where the index takes the value $\chi \in \mathbb{Z}$, then

$$\mathrm{Gr}(V) = \coprod_{\chi \in \mathbb{Z}} \mathrm{Gr}^{\chi}(V)$$

is the decomposition in connected components.

In particular, if V is a finite-dimensional vector space, the points of $\mathrm{Gr}^{\chi}(V)$ are those subspaces of F where $\dim_{\mathbb{C}} F = \chi(F) + \dim_{\mathbb{C}}(V/V_+)$.

Henceforth, we will work with the case $V := \mathbb{C}((z))^{\oplus r}$ and $V_+ := \mathbb{C}[[z]]^{\oplus r}$ $(r \geqslant 0)$ and fix $\chi \in \mathbb{Z}$. Let us recall how the infinite Grassmannian $\mathrm{Gr}^{\chi}(V)$ can be expressed in terms of finite Grassmannians $\mathrm{Gr}^{\chi}(V_{[-m,m)})$ where $V_{[-m,m)} := (z^{-m}V_+)/(z^m V_+)$, $m \in \mathbb{N}$. More general, we introduce the notation $V_{[-m,i)} := (z^{-m}V_+)/(z^i V_+)$, with $m, i \in \mathbb{N}$.

Let us consider

$$\begin{aligned} \widetilde{U}_{m,m} &:= \mathrm{Gr}^{\chi}(V_{[-m,m)}), \\ \widetilde{U}_{m,m+1} &:= \left\{ \begin{matrix} F_{m+1} \in \mathrm{Gr}^{\chi}(V_{[-(m+1),m+1)}) \text{ such that} \\ F_{m+1} + V_{[-m,m+1)} = V_{[-(m+1),m+1)} \text{ and } F_{m+1} \cap V_{[m,m+1)} = (0) \end{matrix} \right\}, \\ \widetilde{U}_{m,i} &:= \Phi_{i-1}^{-1}(\widetilde{U}_{m,i-1}), \qquad i > m+1, \end{aligned}$$

where Φ_m is the rational map $\mathrm{Gr}^\chi(V_{[-(m+1),m+1)}) \longrightarrow \mathrm{Gr}^\chi(V_{[-m,m)})$ defined by

$$\Phi_m(F_{m+1}) := \frac{(F_{m+1} \cap V_{[-m,m+1)}) + V_{[m,m+1)}}{V_{[m,m+1)}}$$

whose domain of definition is the open subscheme $\widetilde{U}_{m,m+1}$.

The schemes $\widetilde{U}_{m,i}$ s fit into the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \widetilde{U}_{0,3} \arrow[r, "\Phi_2"] & \widetilde{U}_{0,2} \arrow[r, "\Phi_1"] & \widetilde{U}_{0,1} \arrow[r, "\Phi_0"] & \mathrm{Gr}^\chi(V_{[-0,0)}) \\
\cdots \arrow[r] & \widetilde{U}_{1,3} \arrow[u, hook] \arrow[r, "\Phi_2"] & \widetilde{U}_{1,2} \arrow[u, hook] \arrow[r, "\Phi_1"] & \mathrm{Gr}^\chi(V_{[-1,1)}) \arrow[u, hook] & \\
\cdots \arrow[r] & \cdots \arrow[u, hook] \arrow[r] & \cdots \arrow[u, hook] & &
\end{tikzcd} \tag{2.1}$$

whose squares are Cartesian. Furthermore, note that $\{(\widetilde{U}_{m,i}, \Phi_{i-1})\}_{i\geqslant m}$ is an inverse system for each m. From proposition 1.5.1 of [EGA-II], one obtains that $\Phi_{i-1} : \widetilde{U}_{m,i} \to \widetilde{U}_{m,i-1}$ is an affine morphism for all i and, hence, the inverse limit $U^m := \lim\limits_{i\geqslant m} \longleftarrow \widetilde{U}_{m,i}$ is an open subscheme of $\mathrm{Gr}^\chi(V)$. Explicitly, one has the following description:

$$U^m = \{F \in \mathrm{Gr}^\chi(V) \text{ s.t. } F + z^{-m}V_+ = V \text{ and } F \cap z^m V_+ = (0)\}.$$

Now, section 2 of [CMP] yields the following.

Theorem 2.1. *For every $m > 0$, U^m is an open subscheme of U^{m+1}. Moreover, the open sets U^m are a covering of* $\mathrm{Gr}^\chi(V)$

$$\mathrm{Gr}^\chi(V) = \bigcup_{m>0} U^m = \bigcup_{m>0} \varprojlim_{i\geqslant m} \widetilde{U}_{m,i}. \tag{2.2}$$

In particular, a subspace $F \in \mathrm{Gr}^\chi(V)$ corresponds to a family of finite-dimensional subspaces, $\{F_{[-i,i)}\}_{i\geqslant m_0}$, where $F_{[-i,i)} \in \widetilde{U}_{m_0,i} \subset \mathrm{Gr}^\chi(V_{[-i,i)})$. Explicitly, given F a family $F_{[-m,i)}$ is constructed as follows:

$$F_{[-m,i)} := \frac{(F \cap z^{-m}V_+) + z^i V_+}{z^i V_+}, \qquad i \geqslant m \geqslant m_0.$$

Conversely, a family $\{F_{m,i}\} \in U^m$ determines a subspace F by the expression

$$F := \bigcup_{m\geqslant m_0} \varprojlim_{i\geqslant m} (F_{m,i} \cap V_{[-m,i)}).$$

In particular, it holds that $F \cap z^{-m}V_+ = \varprojlim\limits_{i\geqslant m} (F_{m,i} \cap V_{[-m,i)})$.

3. Geometric quotient by the action of $\mathrm{Sl}(r, \mathbb{C})$

We will prove that the set of stable points of $\mathrm{Gr}^\chi(V)$ admits a geometric quotient by the action of the group $\mathrm{Sl}(r, \mathbb{C})$. Recall [CMP] has proposed a natural notion of stability for points of the infinite Grassmannian $\mathrm{Gr}^\chi(V)$ with respect to the action of the reductive group $\mathrm{Sl}(r, \mathbb{C})$. That proposal was based on the application of GIT to the finite Grassmannians, $\mathrm{Gr}^\chi(V_{[-m,m)})$. Then, the finite and infinite Grassmannians were related with the help of diagram (2.1) since $\mathrm{Sl}(r, \mathbb{C})$ acts on each term and all maps are equivariant. The following fundamental property was proved.

Proposition 3.1 ([CMP], proposition 3.6)**.** *Let $\widetilde{U}^s_{m,i}$ denote the set of stable points of $\widetilde{U}_{m,i}$ w.r.t. the action of* $\mathrm{Sl}(r, \mathbb{C})$.

It holds that $\Phi_i^{-1}\left(\widetilde{U}^s_{m,i}\right) \subseteq \widetilde{U}^s_{m,i+1}$. In particular, if $\{F_{[-i,i)}\}_{i\geqslant m}$ are associated with $F \in \mathrm{Gr}(V)$ as in equation (2.2) and $F_{[-i_0,i_0)}$ is stable, then $F_{[-i,i)}$ is stable for all $i \geqslant i_0 \geqslant m$.

The definition is as follows.

Definition 3.2 ([CMP], definition 3.7)**.** *Let $F_{[-i,i)} \in U_{m,i} \subset \mathrm{Gr}(V_{[-i,i)})$ be those subspaces associated with a point $F \in \mathrm{Gr}(V)$ by (2.2).*

The point F is (semi)stable for the action of $\mathrm{Sl}(r, \mathbb{C})$ if there exist $m \in \mathbb{N}$ and $i \geqslant m$ such that $F_{[-i,i)}$ is (semi)stable.

We denote the set of the stable and semistable points of $\mathrm{Gr}(V)$ by $\mathrm{Gr}(V)^s$ and $\mathrm{Gr}(V)^{ss}$, respectively.

The above proposition also implies the following.

Proposition 3.3. *Let us denote by $U_{m,m} := \widetilde{U}^s_{m,m}$ and $U_{m,i} := \Phi_{i-1}^{-1}(U_{m,i-1}) \subseteq \widetilde{U}^s_{m,i}$ for each $m \in \mathbb{N}$ and $i > m$.*

It holds that $\{\lim_{i\geqslant m} \longleftarrow U_{m,i} | m > 0\}$ is an increasing sequence of open subsets and that

$$\mathrm{Gr}^\chi(V)^s = \bigcup_{m>0} \varprojlim_{i\geqslant m} U_{m,i}. \tag{3.1}$$

Proof. To begin with, note the following facts; first, $\{(U_{m,i}, \Phi_i)\}_{i\geqslant m}$ is an inverse system for each m; second, there is a diagram

$$\begin{tikzcd} \varprojlim_{i\geq m} U_{m,i} \arrow[r, hook] \arrow[d, hook] & (U^m)^s \arrow[d, hook] \\ \varprojlim_{i\geq m+1} U_{m+1,i} \arrow[r, hook] & (U^{m+1})^s \end{tikzcd}$$

and, finally, that $\mathrm{Gr}^\chi(V)^s = \bigcup_{m>0}(U^m)^s$ (by (2.2)).

Now, let $F \in \mathrm{Gr}^\chi(V)$ be a stable point. Let $\{F_{[-i,i)}\}$ be the subspaces associated with F. From proposition 3.1 there exists m_0 such that $F_{[-i,i)} \in \widetilde{U}^s_{m,i}$ for all m and for all $i > m_0$. Then, $\{F_{[-i,i)}\}$ defines a point of $\lim_{i\geqslant m} \longleftarrow U_{m,i} \subseteq (U^m)^s$ for all $m \geqslant m_0$. And the conclusion follows. □

From theorem 1.10 of [MFK] we know that the open set of stable points of $\mathrm{Gr}^\chi(V_{[-m,m)})$ for the action of $\mathrm{Sl}(r, \mathbb{C})$ does admit a geometric quotient. Furthermore, the open subscheme $U_{m,i}$, which is acted upon by $\mathrm{Sl}(r, \mathbb{C})$ and whose points are stable, also admits a geometric quotient (see 'converse' 1.13 in [MFK]). We denote by $p_{m,i} : U_{m,i} \to Y_{m,i}$ this quotient. The composition $p_{m,i} \circ \Phi_i$ factors through $p_{m,i+1}$, that is

$$\begin{tikzcd} U_{m,i+1} \arrow[r, "\Phi_i"] \arrow[d, "p_{m,i+1}"'] & U_{m,i} \arrow[d, "p_{m,i}"] \\ Y_{m,i+1} \arrow[r, "\Theta_{m,i}"'] & Y_{m,i} \end{tikzcd} \tag{3.2}$$

We have an inverse system $\{(Y_{m,i}, \Theta_{m,i})\}_{i\geqslant m}$ for each $m > 0$. Let $Y^m := \lim\limits_{i\geqslant m} \longleftarrow Y_{m,i}$ and let p^m be the morphism induced by $\{p_{m,i}\}$ between the inverse limits

$$p^m \colon \varprojlim_{i\geqslant m} U_{m,i} \longrightarrow Y^m = \varprojlim_{i\geqslant m} Y_{m,i}.$$

Observe that the family $\{U_{m,i}\}$ fits into a diagram similar to (2.1). Then, applying the properties of inverse limits, one gets a commutative diagram

$$\begin{tikzcd} \varprojlim_{i\geq m} U_{m,i} \arrow[r, hook] \arrow[d, "p^m"'] & \varprojlim_{i\geq m+1} U_{m+1,i} \arrow[d, "p^{m+1}"] \\ Y^m \arrow[r, hook, "\gamma_m"] & Y^{m+1} \end{tikzcd}$$

for each m. Since $\gamma_m : Y^m \hookrightarrow Y^{m+1}$ are open immersions, it makes sense to consider the scheme Y defined by *recollement* of $\{Y^m, \gamma_m\}$; or, in other words

$$Y := \bigcup_{m>0} Y^m. \tag{3.3}$$

Finally, note that there is a map induced by the p^m s

$$p \colon (\mathrm{Gr}^\chi(V))^s = \bigcup_{m>0} \varprojlim_{i\geqslant m} U_{m,i} \longrightarrow Y.$$

Now, it is straightforward to prove the main result of this section.

Theorem 3.4. *The scheme Y is the geometric quotient of* $\mathrm{Gr}^\chi(V)^s$ *by the action of* $\mathrm{Sl}(r, \mathbb{C})$.

Proof. Recall that both $\mathrm{Gr}^\chi(V)^s$ and Y are expressed by a *recollement* of open subschemes (see equations (3.1) and (3.3)). Further, it is straightforward to check that the claim will follow if we prove that Y^m is the geometric quotient of $\lim\limits_{i\geqslant m} \longleftarrow U_{m,i}$ for the action of $\mathrm{Sl}(r, \mathbb{C})$ for all m. In order to this fact we follow the items of definition 0.6 of [MFK].

- The morphism p^m is $\mathrm{Sl}(r, \mathbb{C})$-equivariant. It holds because the morphisms of diagram (3.2) are $\mathrm{Sl}(r, \mathbb{C})$-equivariant.
- *p^m is surjective*. We consider a sequence $\{G_i\}_{i\geqslant m} \in Y^m$. Since $p_{m,i}$ is surjective, we may choose $F_i \in U_{m,i}$ such that $p_{m,i}(F_i) = G_i$. Having in mind that the fiber of G_i is equal to its orbit and that $\Phi_i(F_{i+1}), F_i \in p_{m,i}^{-1}(G_i)$, there exists $g_{i+1} \in \mathrm{Sl}(r, \mathbb{C})$ such that $\Phi_i(g_{i+1}F_{i+1}) = F_i$, for each $i \geqslant m$.
  We now check that the sequence
  $$\{F'_m := F_m, F'_{m+1} := g_{m+1}F_{m+1}, F'_{m+2} := g_{m+1}g_{m+2}F_{m+2}, \ldots\}$$
  is a preimage of $\{G_i\}_{i\geqslant m}$ by p^m.
- For any $W \subseteq Y^m$, W is open if and only if $(p^m)^{-1}(W)$ is open.
  Since p^m is a continuous map, $(p^m)^{-1}(W)$ is open for all open subsets $W \subseteq Y^m$. Let us now show the converse. Let $W \subseteq Y^m$ be a subset such that $(p^m)^{-1}(W)$ is open. From chapter I, section 4, no 4 of [B], we know that any open subset of the inverse limit $\lim\limits_{i\geqslant m} \longleftarrow U_{m,i}$ is of the type $\bigcup\limits_{i\geqslant m} ({}_i)^{-1}(X_i)$, where ${}_i$ are the projections of the limit in each of its factors and X_i are open sets of $U_{m,i}$.

Note that $X_i = p_{m,i}^{-1}(p_{m,i}(X_i))$ and, since $p_{m,i}$ is a geometric quotient, it follows that $p_{m,i}(X_i) \subseteq Y_{m,i}$ is open. Then, the surjectivity of p_m implies that

$$\begin{aligned} W = p^m((p^m)^{-1}(W)) &= p^m\left(\bigcup_{i \geqslant m} (_i)^{-1}(X_i)\right) \\ &= \bigcup_{i \geqslant m} p^m((_i)^{-1}(X_i)) = \bigcup_{i \geqslant m} (j_i)^{-1}(p_{m,i}(X_i)) \end{aligned}$$

and we conclude that $W \subseteq Y^m$ is open.

- It holds that

$$\operatorname{Im}\Gamma^m = \varprojlim_{i \geqslant m} U_{m,i} \times_{Y^m} \varprojlim_{i \geqslant m} U_{m,i},$$

where Γ^m is the morphism

$$\begin{array}{rccc} \Gamma^m : & \mathrm{Sl}(r, \mathbb{C}) \times \varprojlim\limits_{i \geqslant m} U_{m,i} & \to & \varprojlim\limits_{i \geqslant m} U_{m,i} \times \varprojlim\limits_{i \geqslant m} U_{m,i} \\ & (g, F) & \mapsto & (F, gF). \end{array}$$

The inclusion $\subseteq$ is straightforward, so let us prove the reverse one $\supseteq$. Let us take an element

$$(F = \{F_i\}, G = \{G_i\}) \in \varprojlim_{i \geqslant m} U_{m,i} \times_{Y^m} \varprojlim_{i \geqslant m} U_{m,i},$$

therefore, it is verified that $p_{m,i}(F_i) = p_{m,i}(G_i)$ for every $i \geqslant m$. By the properties of $p_{m,i}$, there exist $g_i \in \mathrm{Sl}(r, \mathbb{C})$ for $i \geqslant m$ such that $F_i = g_i G_i$. If we prove that $g_i = g_j$ for all $i, j \geqslant m$, then we obtain $F = gG$ for an element $g \in \mathrm{Sl}(r, \mathbb{C})$ and, thus, $(F, G) \in \operatorname{Im}\Gamma^m$ as was to be shown.

So, let us check that all g_i s are equal. Take $i > j \geqslant m$ arbitrary, and observe that

$$\begin{aligned} g_{i_1} G_{i_1} &= F_{i_1} = \Phi_{i_1} \cdots \Phi_{i_2 - 1} F_{i_2} \\ &= \Phi_{i_1} \cdots \Phi_{i_2 - 1} g_{i_2} G_{i_2} = g_{i_2} G_{i_1}. \end{aligned} \tag{3.4}$$

We conclude that $g_{i_1}^{-1} g_{i_2}, g_{i_2}^{-1} g_{i_1} \in \mathrm{Stab}(G_{i_1})$ where Stab is the stabilizer of a point for the action of a group.

Now, as G_i is a stable point for every $i \geqslant m$, we know that $\mathrm{Stab}(G_i)$ is a finite set (lemma 3.17 of [N]) and we easily have the inclusions

$$\mathrm{Stab}(G_m) \supseteq \mathrm{Stab}(G_{m+1}) \supseteq \cdots.$$

Therefore, there exists $i_0 \geqslant m$ such that $\mathrm{Stab}(G_i) = \mathrm{Stab}(G_{i+1})$ for every $i \geqslant i_0$. For $i < i_0$ we get $g_i^{-1} g_{i_0} \in \mathrm{Stab}(G_i)$ by (3.4). For $i \geqslant i_0$, we obtain $g_i^{-1} g_{i_0} \in \mathrm{Stab}(G_{i_0}) = \mathrm{Stab}(G_i)$ by (3.4) and by the equality of the stabilizers.

So for every $i \geqslant m$, we deduce

$$F_i = g_i G_i = g_i \left(g_i^{-1} g_{i_0} G_i\right) = g_{i_0} G_i.$$

We conclude that $F = g_{i_0} G$ and therefore $(F, G) \in \operatorname{Im}\Gamma^m$.

- The morphism $\mathcal{O}_{Y^m} \to (p^m)_* \mathcal{O}_{\lim\limits_{i \geqslant m} \longleftarrow U_{m,i}}$ induces an isomorphism of $\mathcal{O}_{Y^m}$ and the invariants of $(p^m)^* \mathcal{O}_{\lim\limits_{i \geqslant m} \longleftarrow U_{m,i}}$ under $\mathrm{Sl}(r, \mathbb{C})$. This is true because

$$\mathcal{O}_{Y_m} = \varinjlim_{i \geqslant m} \mathcal{O}_{Y_{m,i}} \simeq \varinjlim_{i \geqslant m} \left(\mathcal{O}_{U_{m,i}}^{\mathrm{Sl}(r,\mathbb{C})}\right) = \left(\mathcal{O}_{\varprojlim\limits_{i \geqslant m} U_{m,i}}\right)^{\mathrm{Sl}(r,\mathbb{C})}.$$

□

4. Moduli of vector bundles with trivialization

In this section, we will study the relation between the moduli spaces of vector bundles with finite trivialization and those with formal trivialization. Then, that relation will be interpreted in terms of Grassmannians. For the construction of the moduli spaces of vector bundles with finite trivialization in terms of the finite Grassmannian we follow [S, AM]. A triple (C, p, t_p) consisting of an irreducible non-singular projective curve over $\mathbb{C}$, a smooth point and an isomorphism of $\mathbb{C}$-algebras $\hat{\mathcal{O}}_p \xrightarrow{\sim} \mathbb{C}[[z]]$ will be fixed from now on. Following [M], we know that there is a $\mathbb{C}$-scheme, $\mathcal{M}_\infty(r, d)$, whose set of rational points is given by

$$\mathcal{M}_\infty(r,d) := \left\{ \begin{array}{c} \text{pairs } (\mathcal{F}, \delta) \text{ s.t. } \mathcal{F} \text{ is a rank r degree d vector bundle} \\ \text{on C and } \delta \text{ is an isomorphism } \hat{\mathcal{F}}_p \xrightarrow{\sim} \hat{\mathcal{O}}_p^{\oplus r} \end{array} \right\},$$

where we write $(\mathcal{F}, \delta) \sim (\mathcal{F}', \delta')$ if and only if there exists an isomorphism of sheaves, $f : \mathcal{F} \xrightarrow{\sim} \mathcal{F}'$ compatible with δ and δ'.

The Krichever map for $\mathcal{M}_\infty(r, d)$ is the scheme homomorphism given by

$$\begin{aligned} \mathcal{K}\colon \mathcal{M}_\infty(r,d) &\longrightarrow \mathrm{Gr}(V, V_+) \\ (\mathcal{F}, \delta) &\longmapsto (t_p \circ \delta)(H^0(C \setminus \{p\}, \mathcal{F})) \end{aligned}$$

with $V := \mathbb{C}((z))^{\oplus r}$ and $V_+ := \mathbb{C}[[z]]^{\oplus r}$. Since this map is a closed immersion, the scheme $\mathcal{M}_\infty(r, d)$ can be thought as a closed subscheme of $\mathrm{Gr}(V)$.

In [S] (see also [AM]) trivializations of finite order have been considered. It has been shown that there exist $\mathbb{C}$-schemes $\mathcal{M}^s_{m,i}(r, d)$, for each pair (i, m) with $i \geqslant m > m_0 := 2g(r + 1)$ (where g is the genus of C), whose set of rational points is given by

$$\mathcal{M}^s_{m,i}(r,d) := \left\{ \begin{array}{c} (\mathcal{F}, \delta_i) \text{ s.t. } \mathcal{F} \text{ is a rank } r \text{ degree } d \text{ stable v.b.} \\ \text{on } C, \delta_i \text{ is a surjection } \mathcal{F} \to (\mathcal{O}_C/\mathcal{O}_C(-ip))^{\oplus r} \\ \text{and } H^0(C, \mathcal{F}(-mp)) = 0,\, H^1(C, \mathcal{F}(mp)) = 0 \end{array} \right\}.$$

The equivalence relation is analogous to the previous one.

The Krichever map for $\mathcal{M}^s_{m,i}(r, d)$ is the scheme homomorphism given by

$$\begin{aligned} \mathcal{K}_{m,i}\colon \mathcal{M}^s_{m,i}(r,d) &\hookrightarrow \mathrm{Gr}^\chi(V_{[-i,i)}) \\ (\mathcal{F}, \delta_i) &\mapsto (t_p \circ \delta_i)(H^0(C \setminus \{p\}, \mathcal{F}(ip))) \end{aligned} \tag{4.1}$$

with $\chi = d + r(1 - g)$ and $V_{[-i,i)} = z^{-i}V_+/z^i V_+$ (see [AM], corollary 2.1).

Let us write down the maps relating these spaces. First, note that $\mathcal{M}^s_{m+1,i}(r, d) \subset \mathcal{M}^s_{m,i}(r, d)$ is an open subscheme for each $i \geqslant m + 1$.

Let us now define an affine and surjective map

$$\Phi_i\colon \mathcal{M}^s_{m,i+1}(r,d) \longrightarrow\!\!\!\!\!\rightarrow \mathcal{M}^s_{m,i}(r,d)$$

which maps $(\mathcal{F}, \delta_{i+1})$ to $(\mathcal{F}, \delta_i)$ where δ_i is given by

$$\mathcal{F} \xrightarrow{\delta_{i+1}} (\mathcal{O}_C/\mathcal{O}_C(-(i+1)p))^{\oplus r} \longrightarrow\!\!\!\!\!\rightarrow (\mathcal{O}_C/\mathcal{O}_C(-ip))^{\oplus r}.$$

Finally, we introduce the rational map

$$\begin{array}{ccc} \mathcal{M}_\infty(r,d) & \longrightarrow & \mathcal{M}^s_{m,i}(r,d) \\ (\mathcal{F}, \delta) & \mapsto & (\mathcal{F}, \delta_i) \end{array} \tag{4.2}$$

whose domain of definition is the open subscheme consisting of those pairs $(\mathcal{F}, \delta)$ such that $\mathcal{F}$ is stable and $H^0(C, \mathcal{F}(-m)) = H^1(C, \mathcal{F}(m)) = 0$. Here δ_i is constructed from δ, since giving an isomorphism $\delta : \mathcal{F} \xrightarrow{\sim} \hat{\mathcal{O}}^{\oplus r}_{C,p}$ is equivalent to giving a compatible family of surjections $\{\delta_i\}$.

Summing up, we have the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{M}^s_{m_0,m_0+3} \arrow[r, "\Phi_{m_0+2}"] & \mathcal{M}^s_{m_0,m_0+2} \arrow[r, "\Phi_{m_0+1}"] & \mathcal{M}^s_{m_0,m_0+1} \arrow[r, "\Phi_{m_0}"] & \mathcal{M}^s_{m_0,m_0} \\
\cdots \arrow[r] & \mathcal{M}^s_{m_0+1,m_0+3} \arrow[r, "\Phi_{m_0+2}"] \arrow[u, hook] & \mathcal{M}^s_{m_0+1,m_0+2} \arrow[r, "\Phi_{m_0+1}"] \arrow[u, hook] & \mathcal{M}^s_{m_0+1,m_0+1} \arrow[u, hook] & \\
\cdots \arrow[r] & \cdots \arrow[r] \arrow[u, hook] & \cdots \arrow[u, hook] & &
\end{tikzcd}$$

From these arguments one deduces the following.

Theorem 4.1. *There is an identification*

$$\{(\mathcal{F},\delta)\in\mathcal{M}_\infty(r,d)\ \text{s.t. } \mathcal{F}\ \text{is stable}\} = \bigcup_{m\geqslant m_0} \varprojlim_{i\geqslant m} \mathcal{M}^s_{m,i}(r,d).$$

The following proposition unveils the relation between this result and theorem 2.1.

Proposition 4.2. *Let $m \geqslant m_0$. The diagram*

$$\begin{tikzcd}
\varprojlim_{i\geq m} \mathcal{M}^s_{m,i}(r,d) \arrow[r] \arrow[d, hook] & U^m = \varprojlim_{i\geq m} U_{m,i} \arrow[d, hook] \\
\{(\mathcal{F},\delta)\in\mathcal{M}_\infty(r,d)\ s.t.\ \mathcal{F}\ is\ stable\} \arrow[r, hook, "\mathcal{K}"] & \mathrm{Gr}^\chi(V)
\end{tikzcd}$$

is Cartesian and the four maps are $\mathrm{Sl}(r,\mathbb{C})$*-equivariant.*

Proof. Given $(\mathcal{F},\delta)\in\mathcal{M}_\infty(r,d)$ such that $\mathcal{F}$ is stable, we know from theorem 2.7 of [M] (see also [O]) that the formal trivialization δ induces canonical isomorphisms

$$F\cap z^m V_+ \simeq H^0(C,\mathcal{F}(-m))$$
$$\frac{V}{F+z^{-m}V_+} \simeq H^1(C,\mathcal{F}(m))$$

for every integer m. Furthermore, the stability of $\mathcal{F}$ implies the stability of $\mathcal{K}((\mathcal{F},\delta))\in\mathrm{Gr}^\chi(V)$ w.r.t. the action of $\mathrm{Sl}(r,\mathbb{C})$ (see [CMP], section 3.2).

Then, the Krichever map (4.1) takes values in $U_{m,i}\subseteq\mathrm{Gr}^\chi(V_{[-i,i)})$; taking inverse limits one gets the arrow on the top row. Now, it is easy to check that

$$\mathcal{K}(\mathcal{F},\delta)\in U^m \iff H^0(C,\mathcal{F}(-m)) = H^1(C,\mathcal{F}(m)) = 0$$

and the conclusion follows. □

Remark 4.3. *As a consequence of these results and those of section 2, it holds that*

$$\mathcal{K}((\mathcal{F},\delta)) = \bigcup_{m\geqslant m_0}\varprojlim_{i\geqslant m}(\mathcal{K}_{m,i}(\mathcal{F},\delta_i)\cap V_{[-m,i)}).$$

Remark 4.4. Let us write down the condition that $(\mathcal{F},\delta),(\mathcal{F}',\delta')\in\mathcal{M}_\infty(r,d)$ have the same image in the quotient. Since their images under p are equal, there exists $g\in\mathrm{Sl}(r,\mathbb{C})$ such that $\mathcal{K}((\mathcal{F},\delta))$ and $g(\mathcal{K}((\mathcal{F}',\delta')))$ coincide or, what is tantamount, there is an isomorphism $\bar{g}:\mathcal{F}\overset{\sim}{\to}\mathcal{F}'$ making commutative the following diagram:

$$\begin{tikzcd}
\mathcal{F} \arrow[r] \arrow[d, "\bar{g}"', "\wr"] & \hat{\mathcal{F}}_p \arrow[r, "\delta", "\sim"'] & \hat{\mathcal{O}}_p^{\oplus r} \arrow[d, "g", "\wr"'] \\
\mathcal{F}' \arrow[r] & \hat{\mathcal{F}}'_p \arrow[r, "\delta'", "\sim"'] & \hat{\mathcal{O}}_p^{\oplus r}
\end{tikzcd}$$

Theorem 4.5. *Let $(\mathcal{F},\delta)\in\mathcal{M}_\infty(r,d)$. Then, $(\mathcal{F},\delta)$ is not stable (as a point of* $\mathrm{Gr}(V)$*) if and only if there exists a vector subbundle $\mathcal{G}\subset\mathcal{F}$ with a formal trivialization, $\gamma:\widehat{\mathcal{G}}_p\overset{\sim}{\to}\widehat{\mathcal{O}}_p^{\oplus l}$ with $l<r$, and $g\in\mathrm{Sl}(r,\mathbb{C})$ such that the following two conditions hold:*

- *$\mu(\mathcal{G})\geqslant\mu(\mathcal{F})$ where μ is the slope of the bundle;*
- *$\delta|_{\widehat{\mathcal{G}}_p}=g\circ\gamma$.*

Proof. Let $F=\mathcal{K}((\mathcal{F},\delta))$. Recall that theorem 3.11 of [CMP] states that $F\in\mathrm{Gr}(k[[z]]^{\oplus r})$ is not stable w.r.t. the action of $\mathrm{Sl}(r,\mathbb{C})$ if and only if there exist $l<r$ and $g\in\mathrm{Sl}(r,\mathbb{C})$ such that $\frac{1}{l}\chi(F\cap gV^l)\geqslant\frac{1}{r}\chi(F)$ where V^l denotes the subspace $k((z))^{\oplus l}\oplus 0\oplus\cdots\oplus 0\subseteq V$.

Let $(\mathcal{F},\delta)$ be a non-stable point and let l and g be as above. Then, from proposition 4.2 of [CMP] (see also proposition 1 of [O]), the subspace $F\cap gV^l$ lies on $\mathrm{Gr}(k((z))^{\oplus l})$ and gives rise to a vector bundle on C, $\mathcal{G}$ endowed with a formal trivialization γ satisfying the conditions of the statement.

The converse can be proved similarly. □

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